# Robustness Radius for Chamberlin-Courant on Restricted Domains

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**Abstract.** The notion of robustness in the context of committee elections was introduced by Bredereck et al. [SAGT 2018] [2] to capture the impact of small changes in the input preference orders, depending on the voting rules used. They show that for certain voting rules, such as Chamberlin-Courant, checking if an election instance is robust, even to the extent of a small constant, is computationally hard. More specifically, it is NP-hard to determine if one swap in any of the votes can change the set of winning committees with respect to the Chamberlin-Courant voting rule. Further, the problem is also W[1]-hard when parameterized by the size of the committee, $k$. We complement this result by suggesting an algorithm that is in XP with respect to $k$. We also show that on nearly-structured profiles, the problem of robustness remains NP-hard. We also address the case of approval ballots, where we show a hardness result analogous to the one established in [2] about rankings and again demonstrate an XP algorithm.

**Keywords:** Robustness radius · Chamberlin-Courant · Single-peaked Single-crossing · NP-hardness

## 1 Introduction

A *voting rule* is a function that maps a collection of preferences over a fixed set of alternatives to a set of winning options, where each option could be one or more alternatives—corresponding, respectively, to the scenarios of single-winner and committee elections. A voting rule is *vulnerable to change* if small perturbations in the input profile can cause its outcome to vary wildly. There have been several notions in the contemporary computational social choice literature that captures the degree of vulnerablity of various voting rules.

A recent exercise in this direction was carried out in [2], where the notion of *robustness radius* was introduced as the minimum number of swaps that was required between consecutive alternatives to change the outcome of a multiwinner voting rule. We note here that we are implicitly assuming that preferences are modeled as linear orders over the alternatives, although the notion of swaps can be defined naturally for the situation where the votes are given by approval


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ballots (each vote indicates the set of approved candidates). In the work of [2], several voting rules are considered, and efficient algorithms were proposed for Robustness Radius for many of these rules. On the other hand, for some voting rules, the problem turned out to be hard: even when the question was to decide if there is *one* swap that influences the outcome. This is the motivation for the present work: we focus on the Chamberlin-Courant voting rule (c.f. Sect. 2 on Preliminaries for the definition), for which Robustness Radius turns out to be intractable, and look for exact algorithms on general profiles and ask if the problem becomes easier to tackle on structured preferences.

*Our Contributions.* Our first contribution is an explicit XP algorithm (recall that a problem is XP parameterized by $k$ if there exists an algorithm which solves it in time $\mathcal{O}(n)^{f(k)}$) for the Robustness Radius problem in the context of the Chamberlin-Courant voting rule. Recall that it is already NP-hard to determine if there exists *one* swap which changes the set of winning committees. Notice that the natural brute-force approach to check if there are at most $r$ swaps which affect the set of winning committees is to simply try all possible ways of executing $r$ swaps and recompute the set of winning committees at every step. This approach, roughly speaking, requires $O((mn)^r \cdot m^k)$ time where $m, n$ are number of candidates and voters (respectively) in the given election instance. We improve this by suggesting an algorithm whose running time can be bounded by $O^\star(m^k)$. We show this result for both the Chamberlin-Courant voting rule with the Borda misrepresentation function as well as for the approval version of the Chamberlin-Courant voting rule. For the latter, we also show that an analogous hardness result holds.

On the other hand, we initiate an exploration of whether the Robustness Radius problem remains hard on structured preferences. We provide some insights on this issue by demonstrating that the problem remains NP-hard on "nearly-structured" profiles. In particular, we show that:

1. Determining if the robustness radius of a profile is one for the $\ell_1$-CC (respectively, $\ell_\infty$-CC) voting rule, with respect to the Borda misrepresentation score, is NP-hard even when the input profiles are restricted to the six-crossing domain[1] (respectively, the four-crossing domain).
2. Determining if the robustness radius of a profile is one for the $\ell_\infty$-CC voting rule, with respect to the Borda misrepresentation score, is NP-hard even when the domain is a four-composite single-peaked domain.

*Related Work.* The notion of robustness is also captured by other closely related notions, such as the margin of victory (MoV) [11] and swap bribery [5]. In the former, the metric of change is the number of voters who need to be influenced, rather than the total number of swaps. On the other hand, in swap bribery, the goal is not to simply influence a change in the set of committees, but to

[1] We refer the reader to the section on Preliminaries for the definition of $\ell$-single-crossing domains. Some definitions and results are deferred to the full version due to lack of space and are marked with a ($\star$).

ensure that a specific committee does or does not win (corresponding to constructive and destructive versions of the problem, respectively). We note that swap bribery has been mostly studied in the context of single-winner voting rules. Observe that any profile that is a non-trivial YES-instance of swap bribery is also a YES-instance of ROBUSTNESS RADIUS with the same budget, but the converse is not necessarily true. Similarly, any profile that is a YES-instance of ROBUSTNESS RADIUS is also a YES-instance of MoV with the same budget, but again the converse need not be true. However, we remark that in the case of the Approval-CC voting rule, the notions of ROBUSTNESS RADIUS and MoV happen to coincide. Robustness has also been studied for single-winner voting rules in earlier work [10].

## 2 Preliminaries

In this section, we introduce some key definitions and establish notation. For a comprehensive introduction, we refer the reader to [1,6].

*Notation.* For a positive integer $\ell$, we denote the set $\{1, \dots, \ell\}$ by $[\ell]$. We first define some general notions related to voting rules. Let $V = \{v_i : i \in [n]\}$ be a set of $n$ *voters* and $C = \{c_j : j \in [m]\}$ be a set of $m$ *candidates*. If not mentioned otherwise, we denote the set of candidates, the set of voters, the number of candidates, and the number of voters by $C$, $V$, $m$, and $n$ respectively.

Every voter $v_i$ has a *preference* $\succ_i$ which is typically a complete order over the set C of candidates (rankings) or a subset of approved candidates (approval ballots). An instance of an election consists of the set of candidates $C$ and the preferences of the voters $V$, usually denoted as $E = (C, V)$. A *multiwinner committee rule* $\mathcal{R}$ is a function that, given an election $E$ and a committee size $k$, outputs a family $\mathcal{R}(E, k)$ consisting winning committees of $k$-sized subsets of $C$.

We now state some definitions in the context of rankings, although we remark that analogous notions exist also in the setting of approval ballots. We say voter $v_i$ prefers a candidate $x \in C$ over another candidate $y \in C$ if $x \succ_i y$. We denote the set of all preferences over C by $\mathcal{L}(C)$. The $n$-tuple $(\succ_i)_{i \in [n]} \in \mathcal{L}(C)^n$ of the preferences of all the voters is called a *profile*. Note that a profile, in general, is a multiset of linear orders. For a subset $M \subseteq [n]$, we call $(\succ_i)_{i \in M}$ a sub-profile of $(\succ_i)_{i \in [n]}$. For a subset of candidates $D \subseteq C$, we use $\mathcal{P}|_D$ to denote the projection of the profile on the candidates in $D$ alone. A *domain* is a set of profiles.

*Chamberlin-Courant for Rankings.* The Chamberlin–Courant voting rule is based on the notion of a *dissatisfaction function* or a *misrepresentation function* (we use these terms interchangeably). This function specifies, for each $i \in [m]$, a voter's dissatisfaction from being represented by candidate she ranks in position $i$. A popular dissatisfaction function is Borda, given by $\alpha_{\mathrm{B}}^{m}(i) = \alpha_{\mathrm{B}}(i) = i - 1$, and this will be our measure of dissatisfaction in the setting of rankings.

We now turn to the notion of an assignment function. Let $k$ be a positive integer. A *$k$-CC-assignment function* for an election $E = (C, V)$ is a mapping

$\Phi\colon V \to C$ such that $\|\Phi(V)\| = k$, where $\|\Phi(V)\|$ denotes the image of $\Phi$. For a given assignment function $\Phi$, we say that voter $v \in V$ is *represented* by candidate $\Phi(v)$ in the chosen committee. There are several ways to measure the quality of an assignment function $\Phi$ with respect to a dissatisfaction function $\alpha$; we use the following:

1. $\ell_1(\Phi, \alpha) = \sum_{i=1,\ldots,n} \alpha(\mathrm{pos}_{v_i}(\Phi(v_i)))$, and
2. $\ell_\infty(\Phi, \alpha) = \max_{i=1,\ldots,n} \alpha(\mathrm{pos}_{v_i}(\Phi(v_i)))$.

Unless specified otherwise, $\alpha$ will be the Borda dissatisfaction function described above. We are now ready to define the Chamberlin-Courant voting rule.

**Definition 1 (Chamberlin-Courant [3]).** *For $\ell \in \{\ell_1, \ell_\infty\}$, the $\ell-$*CC voting rule *is a mapping that takes an election $E = (C, V)$ and a positive integer $k$ with $k \leqslant |C|$ as its input, and returns the images of all the $k$-CC-assignment functions $\Phi$ for $E$ that minimizes $\ell(\Phi, \alpha)$.*

*Chamberlin Courant for Approval Ballots.* Recall that an approval vote $v$ on the set of candidates $C$ is an arbitrary subset $S_v$ of $C$ such that $v$ approves all the candidates in $S_v$. We define the misrepresentation score for $k$-sized commmittee $T$ for an approval voting profile as the number of voters which do not have any of their approved candidates in $T$ (i.e. $T \cap S_v = \phi$). Hence the optimal committees under approval Chamberlin Courant are the committees which maximize the number of voters with at least one approved candidate in the winning committee. This notion of Chamberlin-Courant for the setting of approval ballots was proposed by [8].

*Single Crossing Profiles.* A preference profile is said to belong to the single crossing domain if it admits a permutation of the voters such that for any pair of candidates $a$ and $b$, there is an index $j[(a,b)]$ such that either all voters $v_j$ with $j < j[(a,b)]$ prefer $a$ over $b$ and all voters $v_j$ with $j > j[(a,b)]$ prefer $b$ over $a$, or vice versa. The formal definition is as follows.

**Definition 2 (Single Crossing Domain).** *A profile $\mathcal{P} = (\succ_i)_{i\in[n]}$ of $n$ preferences over a set $C$ of candidates is called a* single crossing profile *if there exists a permutation $\sigma$ of $[n]$ such that, for every pair of distinct candidates $x, y \in C$, whenever we have $x \succ_{\sigma(i)} y$ and $x \succ_{\sigma(j)} y$ for two integers $i$ and $j$ with $1 \leqslant \sigma(i) < \sigma(j) \leqslant n$, we have $x \succ_{\sigma(k)} y$ for every $\sigma(i) \leqslant k \leqslant \sigma(j)$.*

We generalize the notion of single-crossing domains to $r$-single crossing domains in the following natural way (c.f. [9]): for every pair of candidates $(a,b)$, instead of demanding one index where the preferences "switch" from one way to the other, we allow for $r$ such switches. More formally, a profile is $r$-single crossing if for every pair of candidates $a$ and $b$, there exist $r$ indices $j_0[(a,b)], j_1[(a,b)], \ldots j_r[(a,b)], j_{r+1}[(a,b)]$ with $j_0[(a,b)] = 1$ and $j_{r+1}[(a,b)] = n+1$, such that for all $1 \leqslant i \leqslant r+1$, all voters $v_j$ with $j_i[(a,b)] \leqslant j < j_{i+1}[(a,b)]$ are unanimous in their preferences over $a$ and $b$.

*Robustness Radius.* Let $\mathcal{R}$ be a multiwinner voting rule. For the given election $E = (C, V)$, a committee size $k$, and an integer $r$, in the $\mathcal{R}$-Robustness Radius problem we ask if it is possible to obtain an election $E'$ by making at most $r$ swaps of adjacent candidates within the rankings in $E$ (or by introducing or removing at most $r$ candidates from the approval sets of voters in case of approval ballots) so that $\mathcal{R}(E', k) \neq \mathcal{R}(E, k)$.

*Parameterized Complexity.* We occasionally use terminology from parameterized complexity, mainly to describe our results in an appropriate context. A parameterized problem is denoted by a pair $(Q, k) \subseteq \Sigma^* \times \mathbb{N}$. The first component $Q$ is a classical language, and the number $k$ is called the parameter. Such a problem is *fixed–parameter tractable* (FPT) if there exists an algorithm that decides it in time $O(f(k)n^{O(1)})$ on instances of size $n$. On the other hand, a problem is said to belong to the class XP if there exists an algorithm that decides it in time $n^{O(f(k))}$ on instances of size $n$. We refer the reader to [4] for a comprehensive introduction to parameterized algorithms.

## 3 XP Algorithms for Robustness Radius

The Robustness Radius problem for the $\ell_1$-Chamberlin-Courant voting rule with the Borda dissatisfaction function is known to be in FPT when parameterized by either the number of candidates or the number of voters. For the former, the approach involves formulating the problem as an ILP and then using Lenstra's algorithm. In the case of the latter, the algorithm is based on guessing all possible partitions of the voters based on their anticipated representatives and then employing a dynamic programming approach.

In this section, we give a simple but explicit algorithm for the problem which has a XP running time in $k$, the committee size. This complements the W[1]-hardness of the problem when parameterized by $k$ [2]. We establish this result for both when the votes are rankings as well as when they are approval ballots. First, we address the case when the votes are rankings.

**Theorem 1.** *On general profiles comprising of rankings over alternatives,* Robustness Radius *for the $\ell_1$-Chamberlin-Courant voting rule with the Borda dissatisfaction function admits a $O^\star(m^k)$ algorithm, where $m$ is the number of candidates and $k$ is the committee size.*

*Proof.* We first determine the set of all optimal committees of size $k$ in time $O(m^k)$. Suppose there are at least two committees, say $A$ and $B$, that are both optimal. The manner in which this case can be handled is also addressed in [2]. For the sake of completeness, we reproduce the main point here, but in particular we do not address certain edge cases: for example, a slightly different discussion is called for if there are less than $k$ candidates in total occupying the top positions across the votes. We refer the reader to [2] for a more detailed explaination.

Now, note that since $A$ and $B$ are distinct committees, there is at least one voter $v$ whose Chamberlin-Courant representative with respect to $A$ and

$B$ are distinct candidates: say $c_a$ and $c_b$, respectively. Assume, without loss of generality, that $c_a \succ_v c_b$. Note that swapping the candidate $c_b$ so that its rank in the vote $v$ decreases by one results in a new profile where:

1. the dissatisfication score of the committee $B$ is one less than in the original profile, and,
2. the dissatisfication score of the committee $A$ is at least its score in the original profile (indeed; the dissatisfaction score either stays the same or increases if $c_a$ is adjacent to $c_b$ in the vote $v$).

Therefore, when there are at least two optimal committees, it is possible to change the set of winning committees with only one swap, making this situation easy to resolve. We now turn to the case when the input profile admits a unique winning committee $A$. Our overall approach in this case is the following: we "guess" a committee $B$ that belongs to the set of winning committees after $r$ swaps (note that such a committee must exist if we are dealing with a Yes-instance). For a fixed choice of $B$, we determine, greedily, the minimum number swaps required to make $B$ a winning committee. We now turn to a formal description of the algorithm.

Recall that a profile $\mathcal{Q}$ is said to be within $r$ swaps of a profile $\mathcal{P}$ if $\mathcal{Q}$ can be obtained by at most $r$ swaps of consecutive candidates in $\mathcal{P}$. In the following discussion, we say that a committee $B$ is *nearly winning* if there exists a profile $\mathcal{Q}$, within $r$ swaps of $\mathcal{P}$, where $B$ is a winning committee. We refer to $\mathcal{Q}$ as the *witness* for $B$. Note that the existence of a nearly winning committee $B \neq A$ characterizes the Yes-instances. Let $\Delta_{B,A}(\mathcal{P})$ denote the difference between the dissatisfaction scores of the committees $B$ and $A$ with respect to the profile $\mathcal{P}$. We begin by making the following observation.

**Proposition 1.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be two profiles such that $\mathcal{Q}$ can be obtained by making at most $r$ swaps of consecutive candidates in the profile $\mathcal{P}$. Note that:*

$$\Delta_{B,A}(\mathcal{P}) - 2r \leqslant \Delta_{B,A}(\mathcal{Q}) \leqslant \Delta_{B,A}(\mathcal{P}) + 2r.$$

The claim above follows from the fact that if $\mathcal{Q}$ is a profile obtained from $\mathcal{P}$ by one swap of consecutive candidates in some vote of $\mathcal{P}$, then it is easy to see that $\Delta_{B,A}(\mathcal{P}) - 2 \leqslant \Delta_{B,A}(\mathcal{Q}) \leqslant \Delta_{B,A}(\mathcal{P}) + 2$. Note that if $B$ is nearly winning, then $\Delta_{B,A}(Q) \leqslant 0$, where $\mathcal{Q}$ is the witness profile. We now have a case analysis based on $\Delta_{B,A}(\mathcal{P})$.

***Case 1.*** $\Delta_{B,A}(\mathcal{P}) > 2r$**.** In this case, by Proposition 1, we know that in every profile $\mathcal{Q}$ within $r$ swaps of $\mathcal{P}$, $\Delta_{B,A}(\mathcal{Q}) > 0$, which is to say that $B$ will have a greater Borda dissatisfaction score than $A$ in every profile that is $r$ swaps away from the input profile. Therefore, in this case, we reject the choice of $B$ as a potential nearly winning committee.

***Case 2.*** $\Delta_{B,A}(\mathcal{P}) \leqslant r$**.** An analogous argument can be used to see that $B$ is in fact nearly winning in this case. Indeed, any $r$ swaps that improve the ranks of the candidates in $B$ will result in a profile $\mathcal{Q}$ that is within $r$ swaps of $\mathcal{P}$ and where $\Delta_{B,A}(\mathcal{Q}) \leqslant 0$. So, $B$ is either nearly winning with witness profile $\mathcal{Q}$, or $A$ is no longer a winning committee in $\mathcal{Q}$. Therefore, in this situation, we output YES.

***Case 3.*** **$\Delta_{B,A}(\mathcal{P}) = r + s, 1 \leqslant s \leqslant r$.** For a vote $v$, let $A(v)$ and $B(v)$ denote, respectively, the candidates from $A$ and $B$ with the highest rank in the vote $v$. Further, let $d_{B,A}(v)$ denote the difference between the ranks of $B(v)$ and $A(v)$. Let $W \subseteq V$ be the subset of votes for which $d_{B,A}(v) > 0$, and let $w_1, w_2, \ldots$ denote an ordering of the votes in $W$ in increasing order of these differences. We now make the following claim.

**Proposition 2.** *There exists a profile $\mathcal{Q}$ that is $r$ swaps away from $\mathcal{P}$ where $\Delta_{B,A}(\mathcal{Q}) \leqslant 0$ if, and only if:*

$$t := \sum_{i=1}^{s} d_{B,A}(w_i) \leqslant r. \tag{1}$$

*Proof.* In the forward direction, suppose (1) holds. Then perform swaps in the votes $w_1, \ldots, w_s$ so that for any $i \in [s]$, the candidate $B(w_i)$ is promoted to the position just above $A(w_i)$. In other words, each swap involves $B(w_i)$ and in the profile obtained after the swaps, $B(w_i) \succ A(w_i)$ for all $i \in [s]$, and the difference in the ranks of these pairs is exactly one. Note that a total of $t$ swaps are performed to obtain this profile. Denote this profile by $\mathcal{R}$ and note that $\Delta_{B,A}(\mathcal{R}) = r + s - t - s = r - t$ (since the last swap made on each vote $w_i$ reduces the gap between the dissatisfaction scores of the two committees by two). Also, $(r - t)$ is also exactly the number of remaining swaps we can still make, so a witness profile can be obtained using the argument we made in the previous case. The proof of the other direction is deferred to a full version due to lack of space. □

To summarize, our algorithm in this case identifies and sorts the votes in $W$, and returns YES if condition (1) holds, and rejects the choice of $B$ otherwise. Observe that we output NO if no choice of $B$ results in a positive outcome in this case analysis. In terms of the running time, we require $O(m^k)$ time in distinguishing whether we have a unique winning committee or not, and if we are in the former situation, we need $O(m^k)$ time to guess a nearly winning committee. For each choice $B$ of a potential winning committee, we spend time $O(mn \log n)$ in the worst case to determine if $B$ is indeed a nearly winning committee. Therefore, hiding polynomial factors, the overall running time of our algorithm is $O^\star(m^k)$ and this concludes the proof. □

We now turn to the case of approval ballots. First, we show that the robustness radius problem in this setting remains NP-hard even for determining if the robustness radius is one, as was true for the case when the votes were rankings.

**Theorem 2.** ROBUSTNESS RADIUS *for the Approval Chamberlin-Courant voting rule is* NP*-hard, even when the robustness radius is one and each voter approves at most three candidates. It is also* W[2]*-hard parameterized by the size* *of the committee when there are no restrictions on the size of the number of candidates approved by a voter, and the robustness radius is one.*

*Proof.* We reduce from the Hitting Set problem. Note that the NP-hardness in the restricted setting follows from the fact that Hitting Set is already hard for sets of size at most two (recall that this is the Vertex Cover problem), while the W[2]-hardness follows from the fact that Hitting Set is W[2]-hard when parameterized by the size of the hitting set [4] and our reduction will be parameter-preserving with respect to the parameter of committee size.

Let $(U, \mathcal{F}; k)$ be an instance of Hitting Set. Recall that this is a Yes-instance if and only if there exists $S \subseteq U$, with $|S| \leqslant k$ such that $S \cap X \neq \emptyset$ for any $X \in \mathcal{F}$. We construct a profile $\mathcal{P}$ over alternatives $\mathcal{A}$ as follows. Let:

$$\mathcal{A} := \underbrace{\{c_u \mid u \in U\}}_{\mathcal{C}} \ \cup \ \underbrace{\{d_1, \ldots, d_k\}}_{\mathcal{D}}$$

Also, for every $1 \leqslant i \leqslant k$, and for every $X \in \mathcal{F}$, introduce a vote $v(X, i)$ that approves the candidates corresponding to the elements in $X$ along with $d_i$. This completes the construction of the instance. We claim that this instance has a robustness radius of one if and only if $(U, \mathcal{F}; k)$ is a Yes-instance of Hitting Set.

***Forward Direction.*** Suppose $S$ is a hitting set for $(U, \mathcal{F})$ of size $k$. Then the set $C_S := \{c_u \mid u \in S\}$ and $\mathcal{D}$ are two optimal Approval-CC committees with dissatisfaction scores of zero each. Note that removing the candidate $d_1$ from any vote of the form $v(X, 1)$ will lead to a profile where the set of winning committees contains $C_S$ but does not contain $\mathcal{D}$. Hence, the robustness radius is indeed one.

***Reverse Direction.*** For the reverse direction, suppose the profile $\mathcal{P}$ has robustness radius one. We will now argue the existence of a hitting set of size at most $k$. Note that $\mathcal{D}$ is already an optimal committee with respect to $\mathcal{P}$ as it has the best possible Approval-CC dissatisfaction score of zero. Now, suppose $\mathcal{P}$ admits another winning committee $\mathcal{W}$ distinct from $\mathcal{D}$. Then notice that the Approval-CC dissatisfaction score of $\mathcal{W}$ must also be zero, and since there is at least one candidate from $\mathcal{D}$ (say $d_i$) that is not present in $\mathcal{W}$, it is easy to see that the candidates in $\mathcal{C} \cap \mathcal{W}$ form a hitting set for the instance $(U, \mathcal{F}; k)$—indeed, note that every voter in the sub-profile $\{v(X, i) \mid X \in \mathcal{F}\}$ does not approve anyone in $\mathcal{D} \cap \mathcal{W}$, and therefore must approve someone of in $\mathcal{C} \cap \mathcal{W}$, making this a hitting set for $\mathcal{F}$.

Therefore, the interesting case is when $\mathcal{D}$ is the unique winning committee for $\mathcal{P}$. We claim that any other subset of candidates $\mathcal{W}$ of size $k$ has an Approval-CC dissatisfaction score of at least two. This would imply that the robustness radius of $\mathcal{P}$ cannot possibly be one, and therefore there is nothing to prove. To this end, observe that $C_W := \mathcal{W} \cap \mathcal{C}$ is not a hitting set[2] for $\mathcal{F}$: indeed, if $C_W$

[2] Note the slight abuse of terminology here: when referring to $C_W$ as a hitting set, we are referring to the elements of $U$ corresponding to the candidates in $C_W$. As long as this is clear from the context, we will continue to use this convention.

was a hitting set then it is easy to see that $\mathcal{W}$ is also an optimal committee with respect to $\mathcal{P}$, contradicting the case that we are in. Let $X$ denote a set that is not hit by $C_W$. Now, we consider two cases:

*$\mathcal{W}$ Omits Two Candidates from $\mathcal{D}$.* In this case, there are at least two candidates in $\mathcal{D}$—say $d_i$ and $d_j$—who do not belong to $\mathcal{W}$. Then $\mathcal{W}$ earns a dissatisfaction score of one from each of $v(X, i)$ and $v(X, j)$, which makes its dissatisfaction score at least two, as desired.

*$\mathcal{W}$ Omits Exactly One Candidate from $\mathcal{D}$.* In this case, notice that $|C_W| = 1$ and that $C_W$ does not hit at least two sets, say $X$ and $Y$: else $C_W$ along with an arbitrarily chosen element from $X$ and another chosen from $Y$, along with an arbitrary choice of $k - 3$ additional candidates would constitute a winning committee in $\mathcal{P}$ different from $\mathcal{D}$, again contradicting the case that we are in. Therefore, observe that $d_i$ is the candidate from $\mathcal{D}$ that is not present in $\mathcal{W}$, the votes $v(X, i)$ and $v(Y, i)$ contribute one each to the dissatisfaction score of the committee $W$.

Overall, therefore, if $\mathcal{D}$ is the unique winning committee in $\mathcal{P}$, then the robustness radius is greater than one, and there is nothing to prove. This concludes our argument in the reverse direction. □

We now turn to $O^\star(m^k)$ algorithm for Robustness Radius with respect to approval ballots. The general approach is quite analogous to the setting of rankings. However, the notion of swaps is slightly different, and the overall case analysis is, in fact, simpler. Since the main ideas are identical, in the interest of space, we defer a proof of the following claim to a full version of the paper.

**Lemma 1 ($\star$).** *On general profiles comprising of approval ballots over alternatives,* Robustness Radius *for the $\ell_1$-Chamberlin-Courant voting rule with the Borda dissatisfaction function admits a $O^\star(m^k)$ algorithm, where $m$ is the number of candidates and $k$ is the committee size.*

## 4 Hardness for $\ell$-Crossing Profiles

In this section, we explore the complexity of Robustness Radius on nearly-structured preferences. We discover that the problem remains NP-hard parameterized by the size of the committee sought, even on profiles which are 6-crossing even when the robustness radius is one. We note that our overall approach is very similar to the one employed in [2].

**Theorem 3.** *Determining if the robustness radius of a profile is one for the $\ell_1$-CC voting rule, with respect to the Borda misrepresentation score, is* NP*-hard even when the input profiles are restricted to the six-crossing domain.*

*Proof.* We reduce from Independent Set on 3-regular graphs. Let $(G, t)$ be an Independent Set on 3-regular graphs [7]. We construct a profile based on $G$ as follows. Our set of candidates $\mathcal{C}$ is given by:

$$\mathcal{C} := \underbrace{\{c_u \mid u \in V(G)\}}_{V} \cup \underbrace{\{d_1, \ldots, d_h\}}_{\mathcal{D}} \cup \underbrace{\{Z_0, Z_1\}}_{\mathcal{Z}} \cup \underbrace{\{x_1, \ldots, x_{t+1}\}}_{X},$$

where $h$ is a parameter that we will specify in due course. We refer to the candidates in $X$ as the *safe candidates* and $Z_0$ & $Z_1$ are two special candidates. We will use $\tau$ denote a subset of $\Delta$ many unique dummy candidates, where $\Delta := 12nt$. Now we describe the votes. Our voters are divided into three categories as follows:

**Special Candidate Votes:** This group consists of $t+3$ copies of the vote,

$$Z_0 \succ \tau \succ \cdots$$

These votes ensure that every winning committee must include $Z_0$.

**"Safe Committee" Votes:** For each candidate $x_i$ we have $\frac{18t^2}{t+1}$ copies of the vote:

$$v_{x_i} := x_i \succ Z_1 \succ \tau \succ \cdots$$

**Independent Set Votes:** For every edge $\{u, v\}$ in the graph, we introduce $2t$ copies of following two votes:

$$u \succ v \succ Z_0 \succ \tau \succ \cdots$$
$$v \succ u \succ Z_0 \succ \tau \succ \cdots$$

We denote the block of these $4t$ votes by $V_{u,v}$. The intuition for this is to ensure that if some committee has both the endpoints of some edge then the overall misrepresentation will be more than $\Delta$.

The votes described above together constitute our profile $\mathcal{P}$. By fixing an ordering on $\mathcal{C}$ and respecting it on the unspecified votes, it is straightforward to verify that all pairs of candidates cross at most six times in this profile. We note that the candidates corresponding to the vertices cross at most six times because the construction is based on a three regular graph. Define $k = t+2$ and $r = 1$. The $\ell_1$-CC -Robustness Radius instance thus constructed is given by $(\mathcal{C}, \mathcal{P}, k = t+2, r = 1)$. This completes the construction of the instance. We now make some observations about the nature of the optimal committees which will help us argue the equivalence subsequently.

*Possible Winning Committees.* Let $T$ denote the set of candidates corresponding to $t$-sized independent set in $G$ (whenever it exists). We refer to the subset of candidates given by $\{Z_0, x_1, x_2, \ldots, x_{t+1}\}$ as the *safe committee* and denote it by $\mathcal{S}$.

**Lemma 2.** *The constructed profile has a unique winning committee if and only if the graph $G$ has no independent set of size $t$. The safe committee $\mathcal{S}$ has a dissatisfaction score of $\Delta$ and is always a winning committee. If $(G, t)$ is a* Yes *instance, then $\{Z_0, Z_1\} \cup T$ is also an optimal committee, where $T$ denotes an independent set of size $t$ in $G$. Further, any $k$-sized committee not of this form will have dissatisfaction strictly greater than $\Delta + 1$.*

*Proof.* It is easy to see that the dummy candidate will not appear in any optimal committee, since it appears in the top $\Delta$ positions for exactly one vote.

Let us compute the dissatisfaction score for the two proposed committees. For the *safe committee*, we get zero dissatisfaction from the special candidate votes and safe committee votes and we get $8t$ dissatisfaction for each edge which gives us a total dissatisfaction score of $8t \cdot \frac{3n}{2} = 12nt = \Delta$. For the committee based on the independent set, we get zero dissatisfaction from the special candidate votes, $18t^2$ from the *safe committee* votes (one per vote) and $(\frac{3n}{2} - 3t) \cdot 8t + 3t \cdot 2t = 12nt - 18t^2$ from the independent set detector votes. Hence, for both the committees the total dissatisfaction is $\Delta$. It is easy to see that this is the best possible dissatisfaction score that can be achieved by any committee of size $k$.

Note that any optimal winning committee will have candidate $Z_0$ otherwise, one has to pick $k+1$ dummy candidates (to remain optimal), which would exceed the committee size. With $Z_0$ in optimal committee if we intend to choose only few of $x_i's$ then candidate $Z_1$ is forced in the committee. With these constraints, now, we only have two possible structures for any optimal committee. We will analyze both in next part of the proof.

Consider the possible optimal committees which picks $Z_0, Z_1$, few endpoints of edges which are covered twice and the partial independent set (set of vertices which only has one endpoint with given edge). The edges for which both the endpoints are in committee gives zero dissatisfaction, edges for which one endpoint lies in committee gives $2t$ dissatisfaction and edges for which both the endpoints are not in committees gives $8t$ dissatisfaction. Hence, the non-uniformity in dissatisfaction clearly indicates that it is better to cover maximum number of edges by picking one end-point rather than completely losing an edge which causes very high dissatisfaction. So, with the remaining budget for $t$-candidates, the committee with all candidates from independent set will cover maximum edges (to represent by one endpoint) and will cause strictly less dissatisfaction from any other committee by at least $2t$ points.

We now consider a possible winning committee which contains $Z_0, Z_1$, partial independent set and $x_i's$ for the remaining budget. Let's compute the dissatisfaction for this committee. Say we pick $p$ candidates among the $x_i's$ and $(k-2-p) = (t-p)$ candidates from the independent set. The dissatisfaction is:

$$(t + 1 - p) \cdot \frac{18t^2}{t+1} + \left(\frac{3n}{2} - 3(t-p)\right) \cdot 8t + (3 \cdot (t-p) \cdot 2t)$$

which simplifies to: $\Delta + (t-p)\left(\frac{18t^2}{t+1} - 18t\right) + \frac{18t^2}{t+1}$.

For any value of $t$, it is straightforward to verify the above expression has value strictly greater than $\Delta+1$. Hence, committees with this structure will also not be optimal, and this proves the claim. □

Now, we turn to the equivalence of the two instances.

***Forward Direction.*** We need to show that the existence of $t$-sized independent set in the graph implies the existence of one swap of adjacent candidates which changes the set of winning committees for the new election instance. From the above claim we know that when there exist a $t$-sized independent set T, we have two winning committees. In this election instance consider the swap of $Z_1$ with a dummy candidate on right in any of the *safe committee votes*. Now the score for $\{Z_0, Z_1\} \cup T$ is $\Delta+1$ and it's not optimal anymore. Hence, we have changed the set of winning committees. This completes the argument for forward direction.

***Reverse Direction.*** From Lemma 2, we know that unless independent set exists any k-candidate committee other than the *safe committee* has dissatisfaction score strictly greater than $\Delta + 1$. This implies there does not exist any swap which can introduce a new committee in winning committee set (since a single swap can change the score of any committee by at most one) or can knock off *safe committee* from the set. Hence, in this case robustness radius equal to one forces the existence of required independent set (since this is the only committee that can change the set of winning committees). This concludes the proof. □

We remark that an analogous result can be established for the $\ell_\infty$-CC voting rule as well, but exclude the proof due to lack of space.

## 5 Concluding Remarks and Open Problems

We demonstrated XP algorithms for the Robustness Radius problem, when parameterized by the size of the committee, for both the $\ell_1$-CC and the Approval-CC voting rules, using a greedy approach. This complements the known W[1]-hardness of the problem with respect to this parameter. We also explicitly establish the W[2]-hardness of Robustness Radius for the Approval-CC voting rule when parameterized by the size of the committee, even when every voter approves at most three candidates, and when the robustness radius is one. We also established that Robustness Radius for the $\ell_1$-CC and $\ell_\infty$-CC voting rules remains intractable on fairly structured preferences, such as six-crossing profiles.

A natural direction for further thought is if our XP algorithm can be improved to a better running time, especially on structured profiles such as single-peaked or single-crossing domains. A tempting approach is to see if we can exploit the fact that optimal Chamberlin-Courant committees can be computed in polynomial time on these domains. One immediate challenge is the following: if we require our swaps to be such that the resulting profile also remains in the domain that we are working on, then the case when the input profile has multiple winning committees is harder to decide: we can no longer push a committee out of the winning set with one swap, because the said swap may disturb the structure of the profile. We also believe that instead of guessing all possible choices for a nearly winning committee $B$, on structured profiles one might be able to cleverly anticipate the right choice of $B$ without trying all of them.

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