On the Complexity of Winner Verification and Candidate Winner for Multiwinner Voting Rules

Chinmay Sonar[†], Palash Dey[⋆], and Neeldhara Misra[◦]

[†]University of California, Santa Barbara, USA (csonar@cs.ucsb.edu), [⋆]IIT Kharagpur, India (palash.dey@cse.iitkgp.ac.in) [◦]IIT Gandhinagar, India (neeldhara.m@iitgn.ac.in)

Introduction

The Chamberlin-Courant and Monroe rules are fundamental and well-studied rules in the literature of multi-winner elections. The problem of determining if there exists a committee of size k that has a Chamberlin-Courant (respectively, Monroe) dissatisfaction score of at most r is known to be NP-complete. We consider the following natural problems in this setting.

Winner Verification

Input. An election $E = (C, V)$ and a subset S of k candidates.
Question. Is S a winning k-sized CC-committee for the election E?

Candidate Winner

Input. An election $E = (C, V)$, a committee size k, and a candidate $c \in C$.
Question. Does c belong to *some* optimal k-sized committee?

Background

The problem of finding a committee whose misrepresentation is bounded by a given threshold is known to be NP-complete for Chamberlin-Courant and Monroe [1, 2] in the setting of rankings as well as approval ballots.

In a recent development ([3, Theorem 10], improving upon [4, Corollary 3]), it was shown that it is Θ_2^P-hard to determine whether a given candidate belongs to an optimal CC committee in the setting of rankings for the utilitarian method of aggregating misrepresentation scores.

Setting and Definitions

Let V be a set of n *voters* and C be a set of m *candidates*. We denote the set of candidates, the set of voters, the number of candidates, and the number of voters by C, V, m and n, respectively. Every voter v has a *preference* $\succ_v$ which is typically a complete order over the set C of candidates (rankings) or a subset of approved candidates (approval ballots). An instance of an election consists of the set of candidates C and the preferences of the voters V, usually denoted as $E = (C, V)$ with the understanding that the voters in V are identified by their preferences.

We say voter v prefers a candidate $x \in C$ over another candidate $y \in C$ if $x \succ_v y$. For a ranking $\succ$, $\text{pos}_\succ(c)$ is given by one plus the number of candidates ranked above c in $\succ$. We denote the set of all preferences over C by $\mathcal{L}(C)$. The n-tuple $(\succ_v)_{v \in V} \in \mathcal{L}(C)^n$ of the preferences of all the voters is called a *profile*.

The Chamberlin–Courant and Monroe voting rules are based on the notion of a *dissatisfaction* or a *misrepresentation* function. This function specifies, for each $i \in [m]$, a voter's dissatisfaction $\alpha^m(i)$ from being represented by the candidate she ranks in position i. A popular dissatisfaction function is Borda, given by $\alpha^m(i) = i - 1$.

Chamberlin–Courant & Monroe

Let $k \leqslant m$ be a positive integer. A *k-CC-assignment function* for an election $E = (C, V)$ is a mapping $\Phi \colon V \to C$ such that $|\Phi(V)| = k$, where $\Phi(V)$ denotes the image of Φ. For a given assignment function Φ, we say that voter $v \in V$ is *represented* by candidate $\Phi(v)$ in the chosen committee. There are several ways to measure the quality of an assignment function Φ with respect to a dissatisfaction function $\alpha : [m] \longrightarrow \mathbb{R}$; and we will use the following:

- $\ell_1(\Phi, \alpha) = \sum_{v \in V} \alpha(\succ_v(\Phi(v)))$, and
- $\ell_\infty(\Phi, \alpha) = \max_{v \in V} \alpha(\succ_v(\Phi(v)))$.

For $\ell \in \{\ell_1, \ell_\infty\}$, the *$\ell$-CC voting rule* is a mapping that takes an election $E = (C, V)$ and a positive integer k with $k \leqslant |C|$ as its input, and returns the images of all the k-CC-assignment functions Φ for E that minimizes $\ell(\Phi, \alpha)$.

For $\ell \in \{\ell_1, \ell_\infty\}$, the *$\ell$-Monroe voting rule* is a mapping that takes an election $E = (C, V)$ and a positive integer k with $k \leqslant |C|$ as its input, and returns the image of any of the k-Monroe-assignment functions Φ such that $|\Phi^{-1}(c)|$ is either $\frac{n}{k}$ or $\frac{n}{k}$ where $c \in C$ for E that minimizes $\ell(\Phi, \alpha)$.

Main Result — Winner Verification

Winner Verification for Chamberlin-Courant and Monroe is coNP-complete in the setting of approval ballots and rankings. In the latter setting, the result holds for the ℓ_1 and ℓ_∞ misrepresentation functions.

Main Result — Candidate Winner

Candidate Winner for Chamberlin-Courant and Monroe is complete for Θ_2^P in the setting of approval ballots and rankings. In the latter setting, the result holds for the ℓ_1 and ℓ_∞ misrepresentation functions.

Future Work

Investigating the performance of heuristics (by possibly adapting greedy approaches for finding optimal committees and forcing the choice of a desired candidate) would be an interesting direction for complementing our theoretic considerations.

It would be interesting to explore the complexity of the problems we study in the setting of restricted domains. The Winner Verification problems are tractable whenever the naturally associated Winner Determination problem is tractable. In the single-peaked setting, with the ℓ_1-Borda misrepresentation score, the Candidate Winner problem can be resolved by adding several dummy voters who place the desired candidate at the top position, and comparing the optimal CC scores of the original and modified instances. The situation for other restricted domains remains open.

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