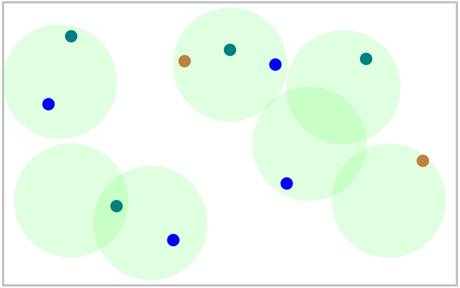
Fair Covering of Points by Balls



Daniel Lokshtanov, Chinmay Sonar, Subhash Suri, Jie Xue
University of California Santa Barbara

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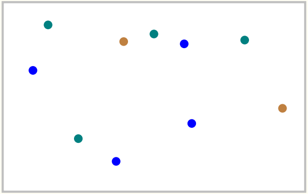

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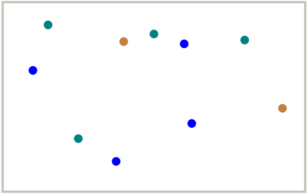

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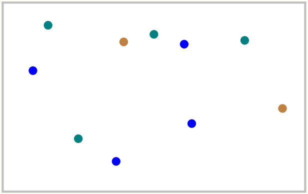

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$c^* \rightarrow$ total number of points covered, then a covering is "fair" iff,

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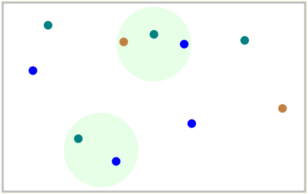

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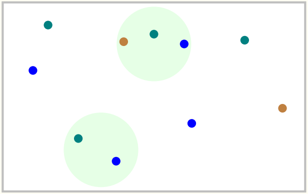

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$$(1-\epsilon)\lfloor \rho_i \cdot c^* \rfloor \leqslant c_i \leqslant (1+\epsilon)\lceil \rho_i \cdot c^* \rceil$$

ϵ-Fair Covering Problem

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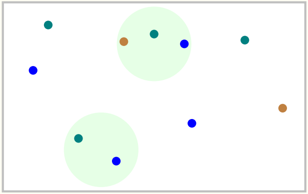

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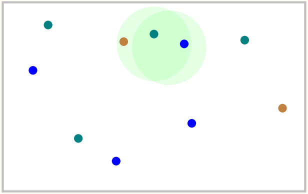

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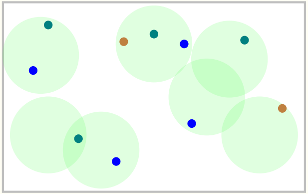

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(We study a novel geometric max cover problem where proportionality implies **both** upper and lower limits for each color)

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Rest of the talk

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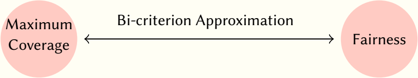

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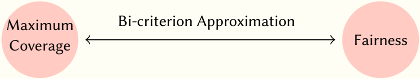

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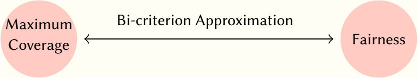

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We compute an ϵ-(r, b)-covering table $\mathcal{C}[r, b]$ that stores the size of an (r, b)-covering $\forall$ valid (r, b) pairs with properties:

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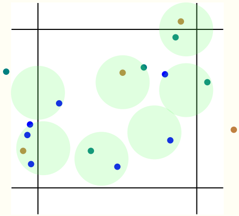

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Apply Shifting Technique and store the best solution in $\mathcal{C}$ after iterating through h^2 Shifting Partitions to get

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Our approach generalizes to $d > 2$ and $t > 2$

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Thank You!