On the Comlpexity of Chamberlin-Courant on Almost Structured Profiles

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Outline

Introduction

Structured Profiles

- In practical settings, elections usually contain some structure

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- Examples include Single-peaked(SP) and Single-crossing(SC)
- When a voting profile has such structure, we refer it to as structured profile
- Many NP-hard voting rules turn out to be polynomially solvable for structured profiles

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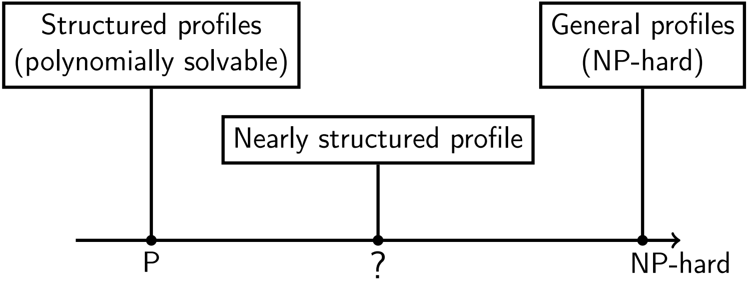

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- A nearly structured profile is a profile that is “close” to admitting structure
- A popular notion of closeness to structure is by deletion of a small part of the profile.
- For example, one might say that a profile is k-close to being single-crossing by voter deletion to mean that there exists a subset S of at most k voters such that the election instance projected on $V \setminus S$ is single-crossing.

Related Work

- The problem of finding deletion sets to single-peaked and single-crossing has been studied [1]

[1] Elkind & Lackner: On detecting nearly structured preference profiles

[2] Chamberlin & Courant: Representative deliberations and representative decisions

[3] Procaccia, Rosenschein & Zohar: Multi-Winner Elections: Complexity of Manipulation, Control, and Winner-Determination

[4] Betzler, Slinko & Uhlmann: On the computation of fully proportional representation

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- Chamberlin-Courant(CC) is one such rule [2]
- CC is NP-hard for the general setting [3] and polynomially solvable for structured profiles [4] [5]

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Results Summary

- We show tractability results for Chamberlin-Courant on profiles that are k candidates/voters away

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Table: Parameterized Complexity of considered multiwinner problems

	SP CC	SP MM[6] CC	SC CC	SC MM CC
Struct[7]	$\mathcal{O}(nm^2)$	$\mathcal{O}(nm)$	$\mathcal{O}(n^2mk)$	$\mathcal{O}(n^2mk)$
VDel	$2^{Rk}\mathcal{O}(nm^2)$	$2^{Rk}\mathcal{O}(nm)$	$2^{Rk}\mathcal{O}(n^2mk)$	$2^{Rk}\mathcal{O}(n^2mk)$
CDel	$2^{k}\mathcal{O}(nm^2)$	$2^{k}\mathcal{O}(nm)$	$2^{k}\mathcal{O}(n^2mk)$	$2^{k}\mathcal{O}(n^2mk)$

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Preliminaries

- Misrepresentation function: For an m-candidate election with votes specified as complete order over set of candidates, a *dissatisfaction function* is given by a non-decreasing function $\alpha\colon [m] \to \mathbb{N}$ with $\alpha(1) = 0$.
  Ex. Borda: $\alpha_B^m(i) = \alpha_B(i) = i - 1$

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  - $\ell_1 = \sum_{i=1,\dots,n} \alpha(\mathrm{pos}_{v_i}(\Phi(v_i)))$, and
  - $\ell_\infty(\Phi) = \max_{i=1,\dots,n} \alpha(\mathrm{pos}_{v_i}(\Phi(v_i)))$

Chamberlin-Courant Voting Rule

Chamberlin Courant-rule

For every family of dissatisfaction functions $\alpha = (\alpha^m)_{m=1}^{\infty}$, and every $\ell \in \{\ell_1, \ell_\infty\}$, the *$\alpha$-$\ell$-CC voting rule* is a mapping that takes an election $E = (C, V)$ and a positive integer k with $k \leqslant |C|$ as its input, and returns a k-CC-assignment function Φ for E that minimizes $\ell(\Phi)$. [a] [b]

[a] Chamberlin & Courant: Representative deliberations and representative decisions

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- Single-crossing profiles

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	3	2	1	0
v_1	a	b	c	d
v_2	b	a	d	c
v_3	c	b	a	d
v_4	c	d	b	a
v_5	d	c	b	a

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Modulators

k-Modulator

A profile is said to have a candidate/voter modulator of size k, if $\exists$ a subset of size at most k candidates/voters such that the restriction of the profile to all but chosen candidates belongs to domain $\mathcal{D}$.

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- Polynomial time algorithm for finding voter modulator and approximation algorithm for candidate modulator have been shown for Single-crossing profiles [8] [9]
- $(\ell, \mathcal{D})$-CC Via χ: denotes aggregation function ℓ over domain D and χ can be candidate or voter modulator

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$(\ell, \mathcal{D})$-CC Extension

- We are given an election $E = (C, V)$, along with the partition of the set of candidates $C = D \uplus G \uplus B$, here G, B represents, partially formed committee and candidates which cannot be part of committee respectively. D represents to be decided through the run of algorithm

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- We are given that the election restricted to (D, V) belongs to domain $\mathcal{D}$
- Objective is to find committee of size b that respects the semantics of (D, G, B) with misrepresentation score at most R

(ℓ, SC)-CC Extension

- Our algorithm builds upon known polynomial time algorithm for Single Crossing profiles [10]

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- Dynamic programming algorithm admits polynomial running time in terms of number of candidates, voters and committee size

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$(\ell, \mathcal{D})$-CC Extension via Candidates

Construction

- Given a k-sized candidate modulator - X

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- If no such Y yields a Yes, return No

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Correctness and Runtime

- Let $C^\star$ be a optimal committee, and $Y^\star$ be $C^\star \cap X$

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- $E = (C, V); (D, G, B)$ is a valid input to $(\ell, \mathcal{D})$-CC Extension and $C^\star$ is a valid solution

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- Runtime(FPT): $2^k q(n, m)$, where $q(n, m)$ is the time required for $(\ell, \mathcal{D})$-CC Extension

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- Setting $G := \bigcup\limits_{v \in X} \mu(v)$, $B := \bigcup\limits_{v \in X} d(v)$ and $D := C \setminus (G \cup B)$ invoke $(\ell, \mathcal{D})$-CC Extension

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- If there exists some guess for which $(\ell, \mathcal{D})$-CC Extension yields Yes, return Yes else return No

$(\ell, \mathcal{D})$-CC Extension via Voters

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- Correctness argument similar to (ℓ, SC)-CC Extension via Candidates case

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Correctness and Runtime

- Correctness argument similar to (ℓ, SC)-CC Extension via Candidates case
- Runtime(XP): $n^k q(n, m)$, where $q(n, m)$ is the time required for $(\ell, \mathcal{D})$-CC Extension
- Open problem: Smarter guessing could yield an FPT runtime, alternatively W-hardness proof could rule out that possibility

Hardness for 3-crossing profiles

Definition of 3-crossing profile

- Another natural way of generalizing the notion of single-crossing profile

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r-crossing Profile

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- $r = 1$ is the familiar single-crossing setting
- Here, we focus on the $r = 3$ i.e. 3-crossing profiles for CC rule

Hardness for 3-crossing profiles

Definition of LSAT

We show a reduction from LSAT to computing an ℓ_∞ -CC committee on 3-crossing profiles with $R \leqslant 2$

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- Variant of SAT where each clause has at most three literals

LSAT is known to be NP-hard [11]

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- Each clause can share at most one literal with another clause

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- Also introduce $(n+1)$ dummy candidates for each variable
- $d[i, j]$ denotes j^{th} dummy candidate for variable x_i

Hardness for 3-crossing profiles

Reduction | Example

LSAT ordering: x_1 , $\overline{x_2}$, x_3 , $\overline{x_1}$, x_4 , x_2 , $\overline{x_4}$, $\overline{x_3}$

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$$\underbrace{(\, x_1 \wedge \overline{x_2} \wedge x_3 \,)}_{C_1} \vee \underbrace{(\, x_3 \wedge \overline{x_1} \wedge x_4 \,)}_{C_2} \vee \underbrace{(\, x_2 \wedge \overline{x_4} \wedge \overline{x_3} \,)}_{C_3}$$

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LSAT ordering: $x_1, \overline{x_2}, x_3, \overline{x_1}, x_4, x_2, \overline{x_4}, \overline{x_3}$

$$(x_1 \wedge \overline{x_2} \wedge x_3) \quad \vee \quad (x_3 \wedge \overline{x_1} \wedge x_4) \quad \vee \quad (x_2 \wedge \overline{x_4} \wedge \overline{x_3})$$

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LSAT ordering: $x_1, \overline{x_2}, x_3, \overline{x_1}, x_4, x_2, \overline{x_4}, \overline{x_3}$

Candidate ordering: $p_1, q_2, p_3, q_1, p_4, p_2, q_4, q_3, d[i,j]$

$$(x_1 \wedge \overline{x_2} \wedge x_3) \quad \vee \quad (x_3 \wedge \overline{x_1} \wedge x_4) \quad \vee \quad (x_2 \wedge \overline{x_4} \wedge \overline{x_3})$$

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LSAT ordering: $x_1, \overline{x_2}, x_3, \overline{x_1}, x_4, x_2, \overline{x_4}, \overline{x_3}$

Candidate ordering: $p_1, q_2, p_3, q_1, p_4, p_2, q_4, q_3, d[i,j]$

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Reduction | Construction

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- Careful case analysis shows that resultant profile is 3-crossing

Thank You !